Mass and Heat Transfer to Single Spheres and Cylinders at Low Reynolds Numbers

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Rates of mass and heat transfer to single spheres and cylinders at low Reynolds numbers are predicted from boundary-layer theory. The velocity distributions which are assumed to exist are those derived from the linearized Navier-Stokes equations by Tomotika and Aoi.

In the case of the sphere the Nusselt number is found to be a function only of the Peclet group when the Stokes streamline function is assumed to apply. Experimental data for mass and heat transfer to single spheres fall 10 to 40% higher than predicted from the theory. Experimental data for heat and mass transfer to single cylinders at large N_{Pe} check the theory.

Curves are also plotted for the efficiency of removal of colloidal particles by combined direct interception and diffusion for both spheres and cylinders.

This paper deals with the theory of mass and heat exchange between single spheres or cylinders and fluids when the relative motion is laminar and natural convection can be neglected. Transfer of this type is of importance in extraction from drops or in the solution of solid particles (continuous phase a liquid) when diameters are smaller than 200 to 500μ. It is of interest in spray drying and aerosol scrubbing (continuous phase a gas) when the drops are smaller than about 100μ in diameter. For cylinders the problem is receiving considerable attention because of its application to aerosol filtration by fibrous filters. In addition, the phenomenon is of importance in the case of low velocity measurements by hot-wire anemometers.

For steady state mass transfer with low concentrations and constant diffusivity, the equation describing diffusion in a moving fluid takes the form

$$u' \frac{\partial c'}{\partial x'} + v' \frac{\partial c'}{\partial y'} + w' \frac{\partial c'}{\partial z'} = D\left[\frac{\partial^2 c'}{\partial x'^2} + \frac{\partial^2 c'}{\partial y'^2} + \frac{\partial^2 c'}{\partial z'^2}\right] \quad (1)$$

A similar expression can be written for heat transfer provided viscous dissipation can be neglected. These expressions apply to any flow regime, but solutions vary depending on the form of the velocity distribution. No general solution appears to exist for laminar flow around spheres or cylinders, although Kronig and Bruijsten (*6*) and Frisch (*4*) have obtained expressions for N_{Nu} for the sphere at very small values of N_{Pe} by use of perturbation methods. For the case of the cylinder Langmuir (*7*) has derived an equation for N_{Nu} by a rough method of approximation.

Heat transfer from flat plates and cylinders at very high Reynolds numbers has been investigated theoretically by use of boundary-layer theory and the Von Karman integral relation. A number of calculations of this type are reported by Eckert (*2*). The approach involves the assumption of the existence of a thermal boundary layer, that is, a limited region of the flow which is affected by the presence of the heated surface. At the edge of the layer the temperature reaches the main-stream value and all the temperature derivatives (with respect to the normal to the surface) vanish. If a temperature distribution is assumed, the heat transfer rate can be calculated by use of the known velocity distribution and a simple heat balance. The velocity profile is usually calculated from momentum–boundary-layer theory, which applies at high Reynolds numbers. The Nusselt number obtained in this way is insensitive to the form of the original temperature distribution, and the theoretical results compare well with experiment up to the point of separation of the boundary layer (*2*). The success of the method together with its simplicity are its principal justification as it lacks rigor in several respects.

In this paper a similar approach is taken but the velocity distribution which is assumed to exist is that derived from the linearized Navier-Stokes equations by Tomotika and Aoi (*9*). Stream functions derived in this way apply for $N_{Re} \leq 5$. In the case of the sphere the method permits evaluation of N_{Nu} over the entire N_{Pe} range in contrast with the perturbation methods (*4*, *6*) which can be used only for small N_{Pe}. For the cylinder a simplified stream function correct only near the surface is used and the results are limited to large N_{Pe}, i.e., to thin boundary layers as in aerosol diffusion and heat and mass transfer in liquids. Again the primary justifications for the attack are the success attending its application and its simplicity.

Few experimental data have been reported in the literature for the low N_{Re} range. Ranz and Marshall (*8*) and Kramers (*5*) have obtained data for mass and heat transfer to spheres. In the high N_{Pe} range, data on heat transfer to cylinders have been obtained by Davis (*1*) and recalculated by Ulsamer (*10*), and data on mass transfer by Dobry and Finn (*3*).

DIFFUSION FROM SINGLE SPHERES

A fluid flows in the positive x direction around a sphere from which material (or heat) diffuses. The steady state has been reached, and resistance to transfer is present only in the fluid. It is assumed that a concentration (or thermal) boundary layer exists around the sphere with a radius b', which is a function only of the angle θ between the radius vector and the positive x axis. It is not necessary that $b' - a$ be small. At the boundary the concentration has attained the main-stream value. A mass balance (mass transfer will be referred to in the rest of the paper although the results are equally applicable to heat transfer) is written for the region between the surface of the sphere and the boundary layer, diffusion being neglected in the θ direction. (The latter is 0 for $N_{Pe} = 0$ and is very small for very large values of N_{Pe}.)

At any angle θ the total amount of material which has diffused from the surface is given by the expression

$$F' = \int_0^{\psi'_b} 2\pi c' \, d\psi' \qquad (2)$$

where $2\pi\psi'$ is the volumetric flow rate of the fluid and $c' = C - C_M$. If no diffusion occurs in the θ direction, the change in F' with θ is due only to the mass diffusing from the surface:

$$-\frac{dF'}{d\theta}\,\delta\theta = -D\left(\frac{\partial c'}{\partial r'}\right)_{r'=a} 2\pi a \sin\theta a \,\delta\theta \qquad (3)$$

or

$$-\frac{d\left[\int_0^{\psi'_b} c'\, d\psi'\right]}{d\theta} = -D\left(\frac{\partial c'}{\partial r'}\right)_{r'=a} a^2 \sin\theta \qquad (4)$$

This equation can also be obtained by partial integration of Equation (1) when diffusion in the θ direction is assumed negligible. For $N_{Re} \ll 1$, ψ' is given by Tomotika and Aoi (*9*) as

$$\psi' = \tfrac{1}{2}Ur'^2\left(1 - \frac{3}{2}\frac{a}{r'} + \frac{1}{2}\frac{a^3}{r'^3}\right)\sin^2\theta \qquad (5)$$

Equation (5) will be recognized as the Stokes stream function for the sphere. When

$$r = \frac{r'}{a}, \qquad \psi = \frac{\psi'}{a^2 U},$$

and

$$c = \frac{C - C_M}{C_0 - C_M} \qquad (6)$$

then

$$-\frac{dF}{d\theta} = -\frac{d\left[\int_0^{\psi_b} c\, d\psi\right]}{d\theta} = -\frac{2}{N_{Pe}}\left(\frac{\partial c}{\partial r}\right)_{r=1}\sin\theta \qquad (7)$$

The boundary conditions which must be satisfied by c and its derivatives are obtained from the definition of the boundary layer and from (1):

at $r = 1$,

$$c = 1, \qquad \frac{\partial\left(r^2 \frac{\partial c}{\partial r}\right)}{\partial r} = 0 \qquad (8)$$

at $r = b(\theta)$,

$$c = 0, \quad \frac{\partial c}{\partial r} = 0, \quad \frac{\partial\left(r^2 \frac{\partial c}{\partial r}\right)}{\partial r} = 0$$

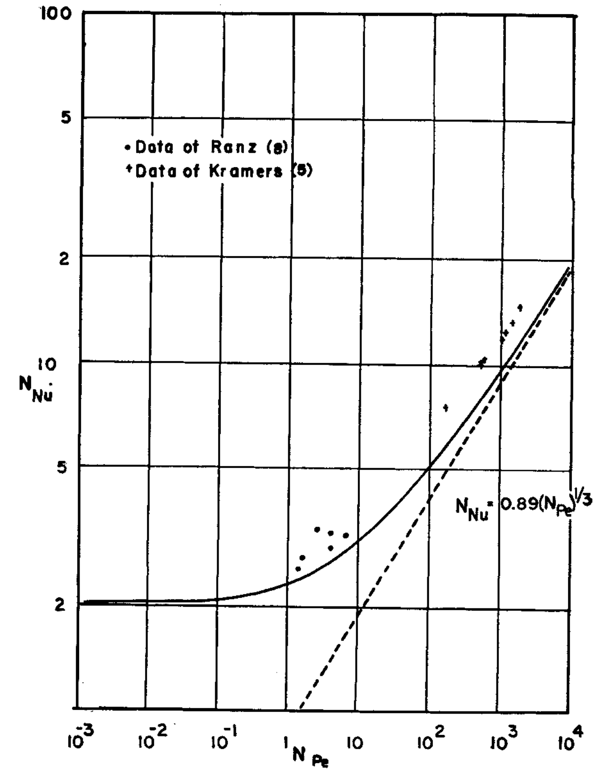

Fig. 1. Transfer to single spheres.

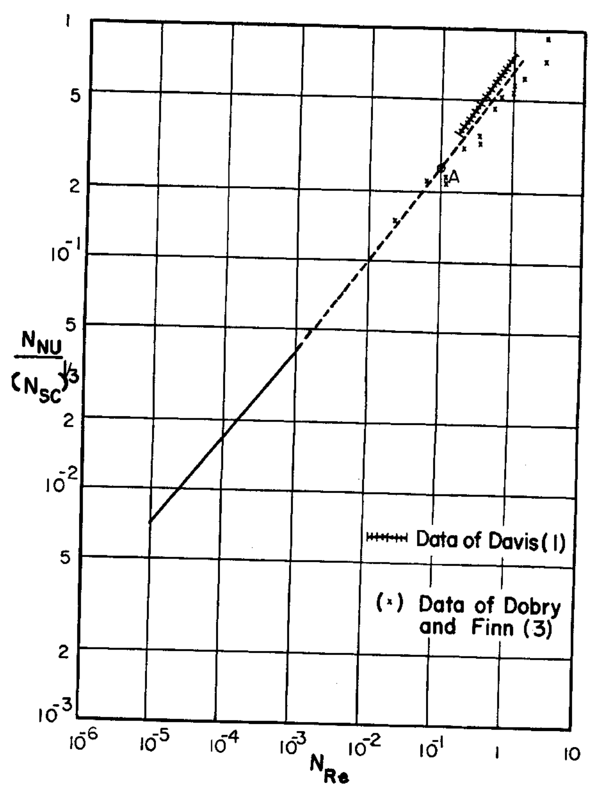

Fig. 2. Transfer to single cylinders.

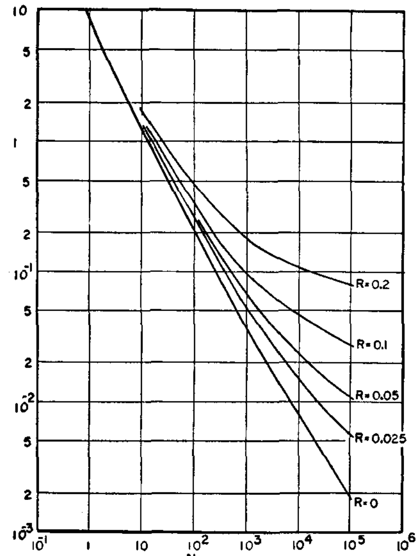

Fig. 3. Removal efficiency for spheres by combined diffusion and direct interception.

In the manner often used in boundary-layer problems, a distribution is assumed for c which satisfies as many boundary conditions as possible. A particularly simple form is

$$c = 1 - \left[\frac{1-\frac{1}{r}}{1-\frac{1}{b}}\right] = 1 - \left[\frac{1-\frac{1}{r}}{h}\right] \quad (9)$$

This expression satisfies all the foregoing conditions with the exception of $\partial c/\partial r = 0$ at $r = b$; it has the additional advantage of reducing to the stagnant distribution when $b = \infty$; viz., $c = 1/r$. Now

$$F = \int_0^{\psi_b} c\, d\psi$$

$$= \int_0^{(c\psi)_{r=b}} d(c\psi) - \int_1^{c_{r=b}} \psi\, dc \quad (10)$$

or with (8)

$$F = -\int_1^0 \psi\, dc \quad (11)$$

Substituting (5) and (9) gives

$$F = \frac{1}{h}\int_1^b \frac{1}{2}\left(1 - \frac{3}{2r} + \frac{1}{2r^3}\right)\sin^2\theta\, dr$$

$$= \frac{1}{h}\left(\frac{b}{2} - \frac{3}{4}\ln b - \frac{1}{8b^2} - \frac{3}{8}\right)\sin^2\theta \quad (12)$$

Substituting in (7) results in

$$\frac{dF}{d\theta} = \frac{d\left[\frac{1}{h}\left(\frac{b}{2} - \frac{3}{4}\ln b - \frac{1}{8b^2} - \frac{3}{8}\right)\sin^2\theta\right]}{d\theta}$$

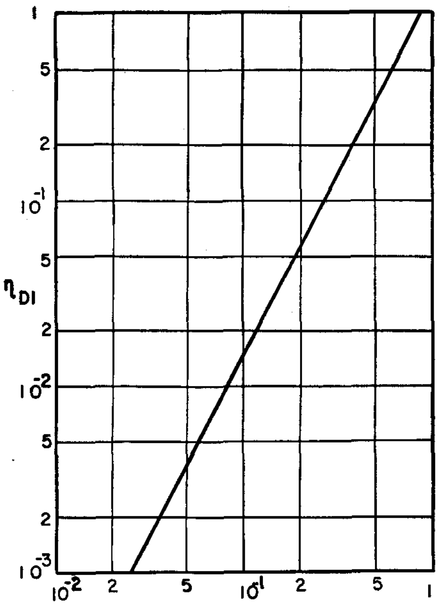

Fig. 4. Removal efficiency by direct interception for single spheres.

$$= \frac{2\sin\theta}{N_{Pe}h} \quad (13)$$

An analytical solution of this equation can be obtained for very large or very small N_{Pe}, but for the intermediate region it is necessary to integrate numerically.

For very large N_{Pe} (very thin boundary layers), F may be expanded as a function of h and then all high-order terms dropped:

$$\frac{F}{\sin^2\theta}$$

$$= \frac{1}{h}\left[\frac{1}{2}(1 + h + h^2 + h^3 + h^4 + \cdots) - \frac{3}{4}\left(h + \frac{h^2}{2} + \frac{h^3}{3} + \frac{h^4}{4} + \cdots\right) - \frac{1}{8} + \frac{1}{4}h - \frac{1}{8}h^2 - \frac{3}{8}\right]$$

$$= \frac{h^2}{4} + \frac{5}{16}h^3 + \cdots \quad (14)$$

or

$$F \cong \frac{h^2}{4}\sin^2\theta$$

Rearranging (13) gives

$$\left(\frac{h\sin\theta}{2}\right)\frac{dF}{d\theta} = -\frac{\sin^2\theta}{N_{Pe}} \quad (15)$$

and integrating between $F = 0$ at $\theta = \pi$ and $F = F_\infty$ at $\theta = 0$ gives

$$F_\infty = \left(\frac{3\pi}{4N_{Pe}}\right)^{2/3} \quad (16)$$

Now

$$N_{Nu} = -\int_0^\pi \left(\frac{\partial c}{\partial r}\right)_{r=1} \sin\theta\, d\theta \quad (17)$$

$$= \int_\pi^0 \frac{N_{Pe}}{2}\frac{dF}{d\theta}\, d\theta = \frac{N_{Pe}F_\infty}{2}$$

$$= 0.89(N_{Pe})^{1/3} \quad (18)$$

If, instead of (9), a polynomial in three terms is used for c, one obtains

$$N_{Nu} = 0.975(N_{Pe})^{1/3} \quad (19)$$

For very small N_{Pe}, $b \longrightarrow \infty$ (stagnant case)

$$F \cong \frac{b}{2}\sin^2\theta \quad (20)$$

Then

$$\frac{d\left(\frac{b}{2}\sin^2\theta\right)}{d\theta} = -\frac{2\sin\theta}{N_{Pe}} \quad (21)$$

whence

$$b = \frac{4}{N_{Pe}(1-\cos\theta)} \quad (22)$$

Substituting in (17) gives

$$N_{Nu} = \frac{4}{N_{Pe}}\ln\left(\frac{1}{1 - N_{Pe}/2}\right) \quad (23)$$

$$= 2\left(1 + \frac{N_{Pe}}{4} + \frac{N_{Pe}^2}{12} + \cdots\right) \quad (24)$$

The first two terms in this expansion were obtained by Kronig and Bruijsten (*6*) in their perturbation solution for small N_{Pe}.

For the intermediate N_{Pe} range it was necessary to integrate (13) numerically. This was done by the method of isoclines, plotting $N_{Pe}F/2$ vs. $\cos\theta$ with lines of constant slope $= 1/h$.

The entire curve is plotted in Figure 1 with the few experimental data available in the low N_{Re} range. In general, the data are from 10 to 40% higher than the theoretical prediction. Three possible explanations for the low trend of the theory are

1. The data were all taken in the range $1 < N_{Re} < 5$, which is somewhat higher than the limit of applicability of (5). At higher N_{Re} small eddies are set up to the rear of the sphere and transfer is increased. (Compare the calculations for the cylinder below.)
2. The theory does not take diffusion in the θ direction into account; this would tend to increase transfer.
3. Similarity hypotheses of the type (9) break down for thick boundary layers.

DIFFUSION FROM THE CYLINDER

The cylinder is treated in a manner analogous to that for the sphere. The fluid flows in the positive x direction and a thermal or concentration boundary layer exists around the cylinder of radius b'. Diffusion in the θ direction is considered negligible, and a mass balance is written for the region between the surface of the sphere and the boundary layer.

At any angle θ the total amount of material which has diffused from the surface is given by the equation

$$F' = \int_0^{\psi'_b} c'\,d\psi' \qquad (25)$$

where ψ' is the volumetric flow rate in the positive x direction. The change in F' with a small change in angle $\delta\theta$ is

$$-\frac{dF'}{d\theta}\,\delta\theta = -\frac{d\left[\int_0^{\psi'_b} c'\,d\psi'\right]}{d\theta}\,\delta\theta$$

$$= -D\left(\frac{\partial c'}{\partial r'}\right)_{r=a} a\,\delta\theta \qquad (26)$$

This equation can also be obtained by partial integration of Equation (1), on the assumption that diffusion in the θ direction is negligible.

For $N_{Re} < 1$, ψ' is given by Tomotika and Aoi (9) as

$$\psi' = -AaU\left[\left(\frac{r'}{a} - \frac{a}{r'} - \frac{2r'}{a}\ln\frac{r'}{a}\right)\sin\theta\right]$$

$$-\frac{N_{Re}}{16}\,aU\left[\left(\frac{r'^2}{a^2} - \frac{a^2}{r'^2}\right.\right.$$

$$\left.\left.-2A\,\frac{r'^2}{a^2}\ln\frac{r'}{a}\right)\sin 2\theta\right] \qquad (27)$$

[It should be noted that the A of (27) is the negative of the A used by Tomotika and Aoi.]

Unlike Equation (5), the stream function for the sphere, this expression applies only in the region fairly near the cylinder, as it does not approach $\psi' = Ur'\sin\theta$ as r' approaches infinity. Thus the treatment which follows is necessarily limited to larger values of N_{Pe}, i.e., to boundary layers contained within the region for which (4) is valid.

If $\psi = \psi'/aU$, (26) may be written in dimensionless terms:

$$-\frac{dF}{d\theta} = -\frac{d\left[\int_0^{\psi_b} c\,d\psi\right]}{d\theta}$$

$$= -\frac{2}{N_{Pe}}\left(\frac{\partial c}{\partial r}\right)_{r=1} \qquad (28)$$

Rearranging (25) gives

$$F = -\int_1^0 \psi\,dc \qquad (29)$$

since the boundary conditions which must be satisfied by c are

at $r = 1$,

$$c = 1, \qquad \frac{\partial\left(r\dfrac{\partial c}{\partial r}\right)}{\partial r} = 0 \qquad (30)$$

at $r = b(\theta)$,

$$c = 0, \qquad \frac{\partial_c}{\partial r} = 0, \qquad \frac{\partial\left(r\dfrac{\partial_c}{\partial_r}\right)}{\partial r} = 0.$$

These conditions except for $\partial c/\partial r = 0$ at $r = b$ are satisfied by

$$c = 1 - \frac{\ln r}{\ln b} \qquad (31)$$

Substituting in Equation (6) yields

$$F = -\frac{1}{\ln b}\int_1^b \left[A\left(1 - \frac{1}{r^2} - 2\ln r\right)\sin\theta\right.$$

$$\left. + \frac{N_{Re}}{16}\left(r - \frac{1}{r^3} - 2Ar\ln r\right)\sin 2\theta\right] dr$$

$$= \frac{1}{\ln b}\left\{A\left(4 + 2b\ln b - 3b - \frac{1}{b}\right)\sin\theta\right.$$

$$+ \frac{N_{Re}}{16}\left[\frac{1}{2}\left(b^2 + \frac{1}{b^2} - 2\right)\right.$$

$$\left.\left. + A\left(b^2\ln b - \frac{b^2}{2} + \frac{1}{2}\right)\right]\sin 2\theta\right\} \qquad (32)$$

In general, (28) must be integrated numerically; however, when N_{Re} is very small and N_{Pe} very large, an analytical solution is possible. If $N_{Re} < 10^{-3}$ the term with the coefficient $N_{Re}/16$ becomes negligible. Also

$$\ln b = \left(1 - \frac{1}{b}\right) + \frac{1}{2}\left(1 - \frac{1}{b}\right)^2$$

$$+ \frac{1}{3}\left(1 - \frac{1}{b}\right)^3 + \cdots$$

$$= (b - 1) + \frac{(b-1)^2}{2} + \frac{(b-1)^3}{3}$$

$$+ \cdots \text{ for } b \text{ near } 1$$

For very large N_{Pe}, i.e., very thin boundary layers, all except the first term in series expansions of $1 - (1/b)$ and $(b - 1)$ may be dropped. Substituting in (28) gives

$$(b-1)\frac{d}{d\theta}\left[\frac{2}{3}A(b-1)^2\sin\theta\right] = -\frac{2}{N_{Pe}} \qquad (33)$$

Integrating between $F = 0$ at $\theta = \pi$ and $F = F_\infty$ at $\theta = 0$ gives

$$F_\infty = \left[\left(\frac{2A}{3}\right)^{1/2}\frac{3\pi^{1/2}}{N_{Pe}}\,\frac{\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{5}{4}\right)}\right]^{2/3} \qquad (34)$$

For the cylinder

$$N_{Nu} = -\frac{1}{2\pi}\int_0^\pi \left(\frac{\partial c}{\partial r}\right)_{r=1} d\theta$$

$$= \int_\pi^0 \frac{N_{Pe}}{\pi}\frac{dF}{d\theta}\,d\theta = \frac{N_{Pe}}{\pi}F_\infty \qquad (35)$$

$$= 1.035(AN_{Pe})^{1/3}$$

$$= 1.035(A\cdot N_{Re})^{1/3}(N_{Sc})^{1/3} \qquad (36)$$

For $N_{Re} > 10^{-3}$ it is necessary to integrate (28) numerically, as the term in (32) with the coefficient $N_{Re}/16$ becomes important. The numerical inte-

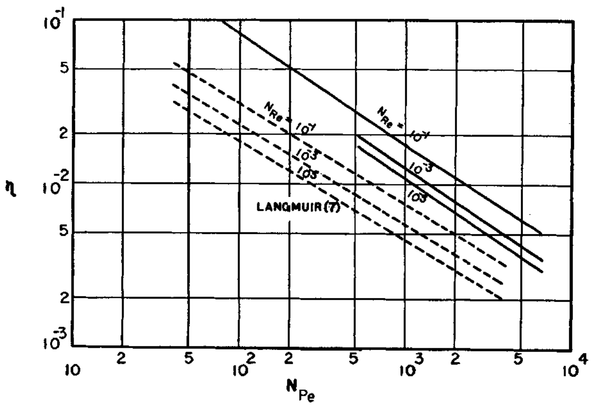

Fig. 5. Removal efficiency by diffusion for cylinders.

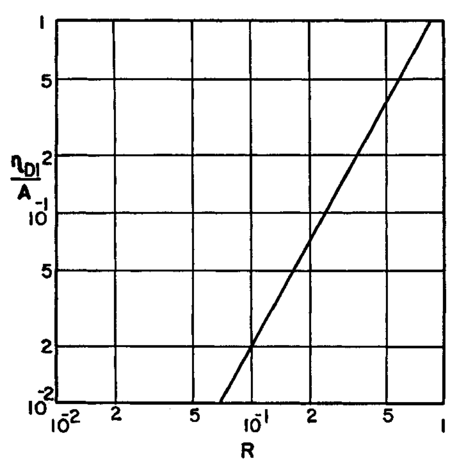

Fig. 6. Removal efficiency by direct interception for single cylinders.

gration was carried out for $N_{Re} = 10^{-1}$ and for this value of N_{Re}, the Nusselt number took the form

$$N_{Nu} = 0.557(N_{Pe})^{1/3} \quad (37)$$

In Figure 2 $N_{Nu}/(N_{Sc})^{1/3}$ is plotted vs. N_{Re}; for $N_{Re} > 10^{-3}$, the curve represents an extrapolation of (36) through the $N_{Re} = 10^{-1}$ point, A, determined by numerical integration. The data of Dobry and Finn (*3*) for mass transfer and the range of the data of Davis (*1*) for heat transfer to oils, as recalculated by Ulsamer (*10*), are also shown. The agreement is good.

The asymptotic solution, Equation (36) is admittedly on the edge of the region applying to problems of present interest. However, its extrapolation through the point determined numerically shows that it is not too much in error over a wider region; moreover, it serves to "hang" the rest of the curve through the numerically evaluated point.

To recapitulate, Equation (36) was derived for $N_{Re} < 10^{-3}$ and $N_{Pe} \gtrsim 10^2$ and (37) was obtained numerically for $N_{Re} = 10^{-1}$ and $N_{Pe} \gtrsim 10$. The large values required for N_{Pe} limit the applicability of Figure 2 to diffusion in liquids and to colloidal-particle transfer.

DIFFUSION AND DIRECT INTERCEPTION OF COLLOIDAL PARTICLES BY THE SPHERE

For the diffusion of a colloidal particle with finite diameter, the concentration falls to zero at $r' = a + a_p$ or $r = 1 + R$ instead of at the surface of the sphere. This is the effect known as *direct interception*. The form assumed for c is

$$c = 1 - \left[\frac{\dfrac{1}{1+R} - \dfrac{1}{r}}{\dfrac{1}{1+R} - \dfrac{1}{b}}\right] \quad (38)$$

and Equation (7) becomes

$$\frac{d}{d\theta}\left[\frac{\sin^2\theta}{\left(\dfrac{1}{1+R}-\dfrac{1}{b}\right)}\int_{1+R}^{b}\frac{1}{2}\left(1-\frac{3}{2r}+\frac{1}{2r^3}\right)dr\right] = -\frac{2\sin\theta}{N_{Pe}\left(\dfrac{1}{1+R}-\dfrac{1}{b}\right)} \quad (39)$$

It would be necessary to solve this equation numerically for each value of R, a tedious process. The combined effect of diffusion and direct interception can, however, be estimated in the following manner.

In Figure 3 the efficiency of removal for pure diffusion ($R = 0$) is plotted vs. N_{Pe}. The removal efficiency is defined as the fraction of the volume swept out by the sphere from which all the diffusing material is removed. For a sphere,

$$\eta = 4\frac{N_{Nu}}{N_{Pe}} \quad (40)$$

For the case of pure interception the efficiency of removal can be calculated from the stream function. The rate of flow of fluid is given by

$$2\pi\psi' = \pi U a^2 r^2 \sin^2\theta\left(1 - \frac{3}{2r} + \frac{1}{2r^3}\right) \quad (41)$$

The point of closest approach of a streamline or particle path to the sphere occurs at $\theta = \pi/2$. Thus the volume of fluid from which particles of radius R are completely removed in unit time is

$$2\pi\psi'(\theta = \pi/2) = \pi U a^2 (1+R)^2 \cdot\left[1 - \frac{3}{2(1+R)} + \frac{1}{2(1+R)^3}\right] \quad (42)$$

and the removal efficiency by interception

$$\eta_{DI} = \frac{\text{volume cleaned}}{\text{total volume swept out}} = (1+R)^2 \cdot\left[1 - \frac{3}{2(1+R)} + \frac{1}{2(1+R)^3}\right] \quad (43)$$

The efficiency η_{DI} is plotted in Figure 4 as a function of R. It is interesting to note that the log-log relationship is linear with

$$\eta_{DI} = 1.45R^2 \quad (44)$$

It is assumed that the removal by diffusion alone is due to the existence of a pseudo particle radius, an inherent diffusion radius, which can be obtained from Figure 4, the plot of η_{DI} vs. R; to each value of η from Figure 3 ($R = 0$) there corresponds a value of R_0, the "diffusion radius," in Figure 4. If to the diffusion radius is added the actual radius, one can then find a new value for η from Figure 4. Values for η as a function of N_{Pe} with R as the parameter were obtained in this way and plotted in Figure 3. Unfortunately, there are no experimental data available with which to check the theory.

DIFFUSION AND DIRECT INTERCEPTION OF COLLOIDAL PARTICLES BY THE CYLINDER

Again the concentration falls to zero at $r = 1 + R$ and the form assumed for c is

$$c = 1 - \frac{\ln\left(\dfrac{1+R}{r}\right)}{\ln\left(\dfrac{1+R}{b}\right)} \quad (45)$$

To determine η as a function of R, it would be necessary to substitute (45) in (28) and then to integrate numerically for each value of N_{Re} and R from $r = 1 + R$ to $r = b(\theta)$. To avoid this tedious process, one estimates the combined removal by interception and diffusion in a manner similar to that in the preceding section:

For the cylinder,

$$\eta = \pi\frac{N_{Nu}}{N_{Pe}} \quad (46)$$

In Figure 5 the removal efficiency η for pure diffusion ($R = 0$) is plotted vs. N_{Pe} with N_{Re} as the parameter. The curve for $N_{Re} = 10^{-1}$ was obtained by

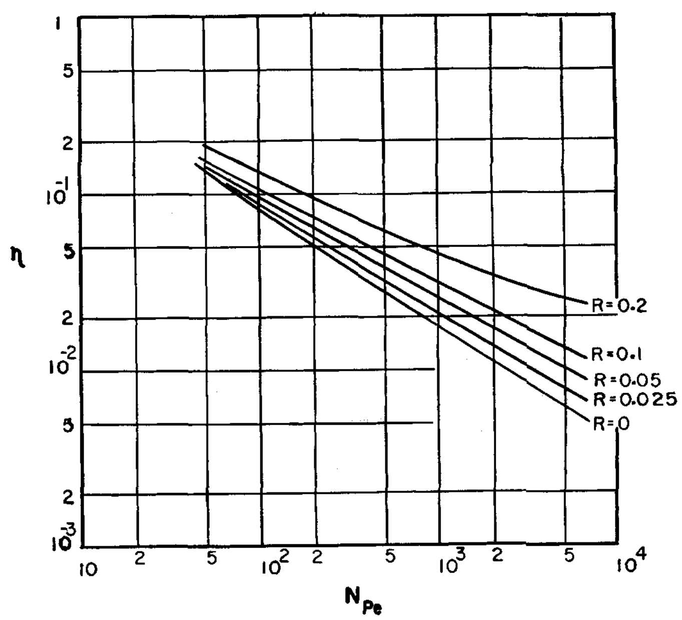

Fig. 7. Removal efficiency by diffusion and direct interception for single cylinders at $N_{Re} = 10^{-1}$.

numerical integration of (26), and the rest of the curves were obtained from the approximate solution (35). Shown also are the theoretical curves of Langmuir (7), which lie below those of the boundary-layer theory and which have a greater slope.

The efficiencies determined by taking the $N_{Re}/16$ term into account are somewhat greater than those calculated from (36) for $N_{Re} = 10^{-1}$. This probably results from the eddies set up behind the cylinder, which are described by the $N_{Re}/16$ term. The efficiency of removal by direct interception,

$$\eta_{DI} = \psi(\theta = \pi/2)$$
$$= -A\left[(1+R) - \frac{1}{(1+R)} - 2(1+R)\ln(1+R)\right] \quad (47)$$

In Figure 6, η_{DI}/A is plotted vs. R, and, as in the case of the sphere, the log-log relationship is linear. The equation for the curve is

$$\eta_{DI} = 1.25AR^{1.82} \quad (48)$$

and for $N_{Re} = 10^{-1}$

$$\eta_{DI} = 0.1455R^{1.82} \quad (49)$$

Also for pure diffusion ($R = 0$) and $N_{Re} = 10^{-1}$, from Figure 5

$$\eta = \frac{1.75}{N_{Pe}^{2/3}} \quad (50)$$

The existence of a pseudo particle radius R_0 is assumed, such that

$$\eta_{DI} = \eta_{R_0(diffusion)} \quad (51)$$

$$\frac{1.75}{N_{Pe}^{2/3}} = 0.1455R_0^{1.82} \quad (52)$$

Solving for R_0 and assuming

$$\eta = 0.1455(R + R_0)^{1.82} \quad (53)$$

one obtains

$$\eta = 0.1455\left[R + \frac{3.96}{N_{Pe}^{0.367}}\right]^{1.82} \quad (54)$$

Values for η obtained in this way are plotted in Figure 7 as a function of N_{Pe} with R as the parameter. This set of curves applies only to $N_{Re} = 10^{-1}$. For values of $N_{Re} < 10^{-3}$ the removal due to the combination of interception and diffusion can be estimated by use of Equation (27):

$$\eta = 1.25A\left[R + \frac{1.69}{(AN_{Pe})^{0.367}}\right]^{1.82} \quad (55)$$

No data available are sufficiently accurate to check the theory; however, it should be noted that the $R = 0$ line (pure diffusion) is equivalent to point A of Figure 2, which agrees well with the data.

CONCLUSIONS

Rates of heat and mass transfer from single spheres or cylinders to fluids in laminar motion can be predicted with some success by means of boundary-layer theory.

Values of N_{Nu} for the sphere can be obtained as a function of the Peclet number alone over the entire N_{Pe} range by numerical integration. For $N_{Pe} > 10^3$ the equations can be solved analytically to give

$$N_{Nu} = 0.89(N_{Pe})^{1/3} \quad (18)$$

For $N_{Pe} \ll 1$ an analytical development gave

$$N_{Nu} = \frac{4}{N_{Pe}} \ln\left(\frac{1}{1 - N_{Pe}/2}\right) \quad (23)$$

For $N_{Pe} < 10^{-1}$, $N_{Nu} \cong 2$, the value for stagnant diffusion.

The few experimental data available in the low N_{Re} range for heat and mass transfer fall from 10 to 40% higher than the theory.

For the case of particles of finite diameter diffusing to a single sphere, efficiencies of removal can be calculated from boundary-layer theory, but a tedious numerical integration would be required for each value of R. Thus an approximate method is used to determine η as a function of R. No experimental data are available to check the theory.

Values of N_{Nu} for the cylinder were obtained as a function of N_{Sc} and N_{Re} for large values of N_{Pe} (thin boundary layers). For $N_{Re} < 10^{-3}$ the equations can be solved analytically to give

$$N_{Nu} = 1.035(AN_{Pe})^{1/3} \quad (36)$$

For $N_{Re} = 10^{-1}$, a numerical integration gave

$$N_{Nu} = 0.557(N_{Pe})^{1/3} \quad (37)$$

With these expressions a plot was made of $N_{Nu}/(N_{Sc})^{1/3}$ as a function of N_{Re}. The theory compared very well with the few experimental data for large N_{Pe}.

For the case of diffusion of particles, efficiencies by combined direct interception and diffusion were estimated by an approximate method. For $N_{Re} = 10^{-1}$

$$\eta = 0.1455\left[R + \frac{3.96}{N_{Pe}^{0.367}}\right]^{1.82} \quad (54)$$

and for $N_{Re} < 10^{-3}$

$$\eta = 1.25A\left[R + \frac{1.69}{(AN_{Pe})^{0.367}}\right]^{1.82} \quad (55)$$

In general, efficiencies determined from boundary-layer theory fall somewhat higher than those calculated by Langmuir (7). No experimental data are available to check the theory.

NOTATION

- a = sphere or cylinder radius, cm. or μ
- a_p = colloidal-particle radius, cm. or μ
- A = $\dfrac{1}{2(2.00223 - \ln N_{Re})}$
- b = b'/a, dimensionless
- b' = boundary-layer radius, cm. or μ
- c = $(C - C_M)/(C_0 - C_M)$, dimensionless
- c' = $C - C_M$, g., g. moles, or particles/cc.
- C = concentration, g., g. moles, or particles/cc.
- C_M = main-stream concentration, g., g. moles, or particles/cc.
- C_0 = concentration at surface of cylinder or sphere, g., g. moles, or particles/cc.
- D = diffusivity, sq. cm./sec.
- F = quantity of diffusing material, dimensionless
- F' = quantity of diffusing material, g., g. moles, or particles
- h = $1 - 1/b$
- k = mass transfer coefficient, cm./sec.
- N_{Nu} = Nusselt number, $2ka/D$, dimensionless
- N_{Pe} = Peclet number, $2aU/D$, dimensionless
- N_{Re} = Reynolds number, $2aU\rho/\mu$, dimensionless
- N_{Sc} = Schmidt number, $\mu/\rho D$, dimensionless
- r = r'/a, dimensionless
- r' = radial distance, cm. or μ
- R = a_p/a, dimensionless
- R_0 = diffusion radius, dimensionless
- u' = velocity in x direction cm./sec.
- U = main-stream velocity, cm./sec.
- v' = velocity in y direction, cm./sec.
- w' = velocity in z direction, cm./sec.
- η = removal efficiency by diffusion and direct interception, dimensionless
- η_{DI} = removal efficiency by direct interception, dimensionless
- ρ = fluid density, g./cc.
- θ = angular displacement from positive x axis
- μ = fluid viscosity, g./(cm.)(sec.)
- ψ = stream function, dimensionless
- ψ' = stream function, cc./sec.
- ψ_b = stream function at boundary, dimensionless
- ψ_b' = stream function at boundary, cc./sec.

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Presented at A.I.Ch.E. Detroit meeting